$$-\left[\left(\frac{L-x_o}{2\pi}\right)\sin\left(\frac{w}{w_c}\pi\right)\right]\Bigg\}$$

Finally, to facilitate the use of this equation it is reduced to a graphical solution.

NOTATION

A = parameter defined by Equation (23)
A_H = heat transfer area, sq. ft.
A_M = mass transfer area, sq. ft.
B = parameter defined by Equation (24)
C = vapor concentration, lb./cu. ft.
C_P = specific heat at constant pressure, B.t.u./(lb.) (°F.)
D = volumetric diffusivity, sq. ft./hr.
k_e = effective thermal conductivity of solid, B.t.u./(hr.) (ft.) (°F.)
k_L = thermal conductivity of liquid, B.t.u./(hr.) (ft.) (°F.)
L = total depth of bed, ft.
M_L = molecular weight of liquid
P = vapor pressure, lb./sq. in.
p = partial pressure, lb./sq. in.
R = universal gas constant, B.t.u./lb. mole °F.
R_H = resistance to heat transfer, sq. ft. hr. °F./B.t.u.
S = pore liquid content or saturation, cu. ft. liquid/cu. ft. of void space in wet portion of bed
T = absolute temperature, °R.
t = temperature, °F.
V = bed volume, cu. ft.
W = total weight of bed, lb.
w = free moisture content, lb. liquid/lb. bed
x = depth of liquid surface from bed surface, ft.
X = parameter defined by Equation (25)

Greek Letters

ϵ = void fraction, volume voids/volume bed
λ = latent heat of vaporization, B.t.u./lb.
ρ_B = density of dry bed, lb./cu. ft.
ρ_L = density of liquid, lb./cu. ft.
τ = time constant for the system, hr.

Subscripts

o = property evaluated at initial bed temperature
wb = property evaluated at wet-bulb temperature of the air
pwb = property evaluated at pseudo-wet-bulb temperature
a = property evaluated at air temperature
c = indicates the property value at critical moisture content
$C.R.$ = property evaluated at the constant rate of drying condition

Superscripts

— = average property for entire bed

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Manuscript received August 18, 1959; revision received October 30, 1959; paper accepted November 17, 1959.

Mass Transfer in Laminar—Boundary-Layer Flows with Finite Interfacial Velocities

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Asymptotic expressions are presented in this paper for the mass transfer Nusselt number in both forced- and free-convection laminar-boundary-layer flows with large interfacial velocities directed toward the surface. The analysis is valid for arbitrary surface geometries and includes transfer in a variable properties fluid. It is shown, in addition, how these asymptotic formulas may be used in conjunction with the Nusselt number for zero interfacial velocities to estimate the rate of mass transfer for the intermediate regions.

In mass transfer operations the surface past which the fluid moves has the additional property of acting as a source or a sink of material to the flowing stream. Strictly speaking, therefore, mass and heat transfer are not completely analogous phenomena, even under seemingly identical flow configurations; in the formal case the interfacial velocity is usually finite, rather than identically zero, owing to the material exchange on the surface. In particular, the breakdown of the analogy between mass and heat transfer is most pronounced when the mole fraction of the diffusing species is almost zero near the surface and almost unity in the main stream (or vice versa), for then the normal component of the hydrodynamic velocity at the surface is no longer negligible and must be included in the analysis.

Therefore, owing to the importance of mass transfer in a great many operations—adsorption, sweat cooling, heterogeneous catalysis, to name but a few—it is of some interest to examine under what conditions the rate of mass transfer is affected by the finite interfacial velocity at the surface and how this effect can be predicted in those cases where it appears to be of importance.

The purpose of this article is then to attempt to answer these questions rigorously. The analysis will be limited to forced- and free-convection mass transfer in boundary-layer flows, under isothermal conditions, where the interfacial velocity is directed toward the surface. Further studies dealing with other aspects of this general problem will be considered in the future.

BASIC EQUATIONS FOR FORCED CONVECTION

For the flow of a mixture containing an inert and the diffusing component past the arbitrary two-dimensional surface of Figure 1, the problem is how to predict the rate of transfer of the diffusing substance from the bulk of the fluid to the surface, with which it may react. It will be shown below that for hydrodynamic conditions of the so-called "laminar–boundary-layer" type a simple solution to the problem may be obtained.

This phenomenon is governed by the basic laws of fluid mechanics, the Navier-Stokes and continuity equations, and in addition by the diffusion equation for convective mass transfer. This paper will deal only with laminar systems for which the standard boundary-layer simplifications (*5, 2, 3*) may be used. The basic equations are therefore the momentum balance

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = U_i \frac{dU_i}{dx_1} + \frac{\partial^2 u_1}{\partial y_1^2} \tag{1}$$

the mass balance

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 \tag{2}$$

and the diffusion equation

$$u_1 \frac{\partial \theta}{\partial x_1} + v_1 \frac{\partial \theta}{\partial y_1} = \frac{1}{N_{Sc}} \frac{\partial^2 \theta}{\partial y_1^2} \tag{3}$$

where, for the moment, constant fluid properties have been assumed.

This system of equations is subject to the boundary conditions:

$$\text{At } y_1 = 0, \qquad u_1 = 0, \qquad \theta = 0 \tag{4a}$$

$$\text{At } y_1 = \infty, \qquad u_1 = U_i, \qquad \theta = 1 \tag{4b}$$

$$\text{At } x_1 = 0 \qquad u_1 = U_i, \qquad \theta = 1 \tag{4c}$$

$$\text{At } y_1 = 0 \qquad v_1 = -\frac{n}{N_{Sc}} \left(\frac{\partial \theta}{\partial y_1} \right) \tag{4d}$$

where

$$n = \frac{W_\infty - W_i}{1 - W_i}$$

(for mass transfer into the surface $W_i < W_\infty$ and therefore $0 < n < 1$)

The last restriction is obtained from the realization that the transfer rate of the inert into the surface is zero (*1, 3, 7*).

It is precisely this last boundary condition, Equation (4*d*), which is responsible for the breakdown of the analogy between heat and gas-phase mass transfer. When $n \to 0$, $v_1 = 0$ at the surface, and the velocity distribution may be calculated independently of the diffusion equation, which is of course what may be done for heat transfer in a constant properties fluid.

It is clear, however, that when n is appreciably different from zero, the momentum and diffusion equations are coupled and cannot, be solved independently of one another. In particular, if one recalls how difficult it is to calculate the velocity and the concentration distributions past a surface with a negligible interfacial velocity, it is rather surprising indeed that, when $n \to 1$, the present problem may be solved in a closed form for an arbitrary surface geometry.

A rigorous solution of Equations (1) to (4) is of course in general an impossible task. Exact analytic solutions are hard to come by unless one is dealing with wedgelike surfaces for which the standard similarity transformations of boundary-layer theory may be applied. On the other hand, approximate momentum integral methods, the von Kàrmàn-Pohlhausen technique for example, are cumbersome to use and may be inaccurate. A different approach is therefore needed.

It should be realized at this point now that the complete solution of Equations (1) to (4) is of little interest. What is required is not the complete velocity and mass fraction profiles but an expression for the local rate of mass transfer into the surface as a function of the parameters of the system: N_{Re}, N_{Sc}, n, and the surface geometry. This is normally accomplished by relating the Nusselt number to the above-mentioned variables, where

$$N_{Nu} \equiv -L \left(\frac{\partial \theta}{\partial y} \right)_{y=0} = -\sqrt{N_{Re}} \left(\frac{\partial \theta}{\partial y_1} \right)_{y_1=0} \tag{5}$$

and since it is clear from Equations (1) to (4) that the quantity $(\partial \theta / \partial y_1)_{y_1=0}$ is a function only of x_1, n, and N_{Sc} for a given surface,

$$\frac{N_{Nu}}{\sqrt{N_{Re}}} = F(N_{Sc}, x_1, n) \tag{6}$$

What happens to the function F as $n \to 0$ will not be considered here. This is the classical problem of mass transfer with zero interfacial velocity which has been studied in great detail by many investigators. Considerable information exists therefore about $F(N_{Sc}, x_1, 0)$ for various surface geometries, and a fair number of reliable approximate methods have been proposed for predicting this function.

It is clear then that since the behavior of F for $n \to 0$ may be assumed to be known, what is primarily needed is its asymptotic form for $n \to 1$. For it is found from experience with analogous physical setups that if the behavior of a phenomenon can be predicted

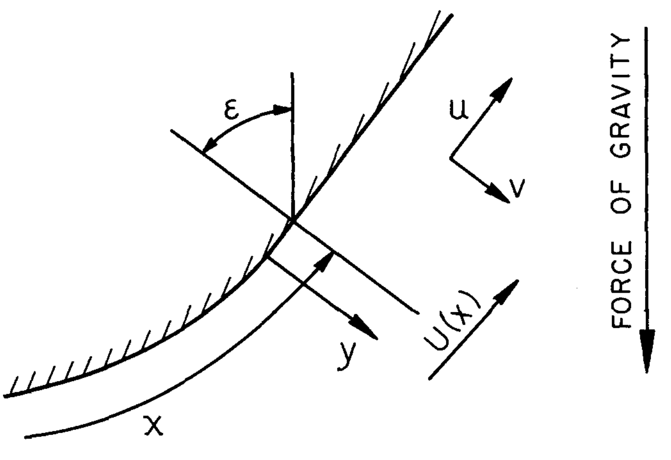

Fig. 1. The position directions of the coordinates *x* and *y*, the velocities *u* and *v*, and the angle ζ.

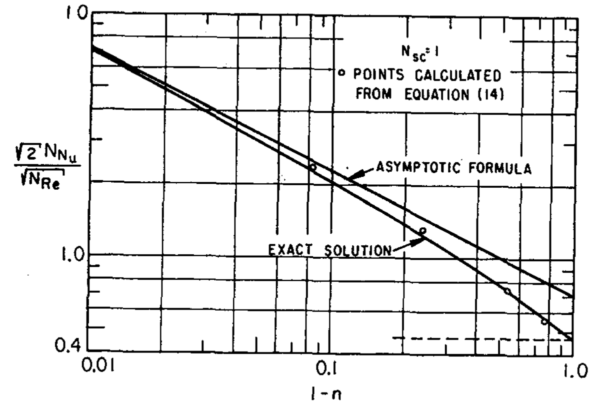

Fig. 2. Mass transfer for flow past a flat plate.

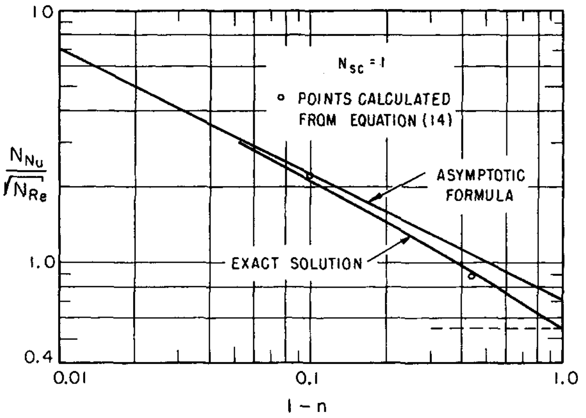

Fig. 3. Mass transfer for stagnation flow.

under the asymptotic conditions where a certain characteristic dimensionless group of the system may take on extreme values (very large or very small), it is a relatively easy matter to describe the process for intermediate values of this dimensionless group by the use of an interpolation formula. One is led to believe therefore that if the limiting form of $F(N_{Sc}, x_1, n)$ for $n \to 1$ were to be discovered, a simple interpolation between its two asymptotes would yield a reliable estimate of this function for all values of n. Experience has taught the author that this approach is far quicker, and the results thus obtained are at least as accurate as those arrived at by momentum integral methods.

ASYMPTOTIC SOLUTION FOR FORCED CONVECTION AS $n \to 1$

Constant property fluids

Equations (1) to (4) admit of a simple solution when the parameter n is near unity. Following a well-established tradition, one begins by attempting to discover a similarity transformation which will reduce the partial differential equations into ordinary differential equations. This can fortunately be accomplished without too much difficulty.

Let

$$u_1 = U_i(x_1)F'(\eta) \text{ and } \theta = \theta(\eta) \quad (5)$$

where

$$\eta \equiv \frac{y_1 U_i}{\sqrt{2\int_0^{x_1} U_i dx}} \quad (6)$$

which when substituted into the original equations reduce them to

$$F''' + FF'' = [(F')^2 - 1]\frac{2U'_i}{U_i^2}\int_0^{x_1} U_i dx \quad (7a)$$

and

$$\theta'' + N_{Sc}F\theta' = 0 \quad (7b)$$

with the boundary conditions

$$\text{At } \eta = 0 \quad F = \frac{n}{N_{Sc}}\theta', \; F' = 0, \; \theta = 0 \quad (8)$$

$$\text{At } \eta = \infty \quad F' = 1, \; \theta = 1$$

At this point, one readily perceives from Equation (7a) that, strictly speaking, similarity solutions for the original system of equations which would also satisfy all the imposed boundary conditions cannot generally be obtained unless the surface geometry is such that $\frac{2U'_i}{U_i^2}\int_0^{x_1} U_i dx$ = a constant for all x_1 a condition which is satisfied only if U_i is of the form

$$U_i = x_1^m$$

This, as is well known (5), is the potential flow distribution past a wedge, and therefore similarity solutions for arbitrary values of n exist only for the wedge type of flow. The author will presently show, however, that as $n \to 1$, the right-hand side of Equation (7a) may be neglected and the similarity transformation given by Equation (6) may be applied to any surface geometry.

Physical intuition suggests now that as $n \to 1$ the thickness of the boundary layer will become vanishingly small and the rate of mass transfer, proportional to θ' at the surface, will increase beyond bound. Therefore let

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} \equiv b \quad (9)$$

where $b \to \infty$ as $n \to 1$, and define

$$z \equiv b\eta \quad (10a)$$

$$F \equiv \frac{nb}{N_{Sc}} + \frac{1}{b}\phi(z) \quad (10b)$$

$$\theta = \theta(z) \quad (10c)$$

which, when substituted into Equations (7), becomes

$$\phi''' + \frac{n}{N_{Sc}}\phi'' = \frac{1}{b^2}\left\{-\phi\phi'' + [(F')^2 - 1]\frac{2U'_i}{U_i^2}\int_0^{x_1} U_i dx\right\}$$

$$\to 0 \text{ as } b \to \infty \quad (11a)$$

and

$$\theta'' + N_{Sc}\left[\frac{n}{N_{Sc}} + \frac{\phi}{b^2}\right]\theta' = 0 \quad (11b)$$

with the boundary conditions

$$\text{At } z = 0 \quad \phi = 0, \; \phi' = 0, \; \theta = 0, \; \theta' = 1 \quad (12)$$

$$\text{At } z = \infty \quad \phi' = 1, \; \theta = 1$$

Two things are immediately apparent: (1) as $n \to 1$, and therefore $b \to \infty$, the similarity transformation given by Equation (6) does indeed apply to surfaces of arbitrary geometry, since the right-hand side of Equation (11a) vanishes in the limit, and (2) the solution of Equations (11) and (12) is a classical eigenvalue problem for the unknown quantity b.

It can be shown now by direct integration, with careful use made of the fact that $b >> 1$, that

$$b = \sqrt{\frac{1}{1-n}\,\frac{N_{Sc}}{1+N_{Sc}}}$$

and therefore

$$N_{Nu} = \frac{U_i}{\sqrt{\int_0^{x_1} U_i dx}}\sqrt{\frac{N_{Re}\cdot N_{Sc}}{2(1-n)(1+N_{Sc})}}\{1 + O(1-n)\} \quad (13)$$

as $n \to 1$ for arbitrary two-dimensional surfaces. An identical result, restricted however to a flat-plate geometry with $U_i \equiv 1$, was arrived at by Spalding (7) by means of a momentum-integral method.

As was pointed out earlier, Equations (1) to (4) may be reduced by a similarity transformation for all values of n only when the surface geometry is wedgelike. The resulting ordinary differential equations have been solved numerically for the velocity distribution past two simple surfaces, the flat plate and the stagnation region, when the normal interfacial-velocity component is specified *a priori* (6, 4). The corresponding mass transfer problem has however not been worked out in such detail except for Schmidt numbers of 0.7 and 1.0 (2). Figures 2 and 3 show how the exact values for $N_{Nu}/\sqrt{N_{Re}}$ compare with the asymptotic formula, Equation (13), for the flow past a flat plate and in the stagnation region for $N_{Sc} = 1$. The curves labeled *exact* have been drawn on the basis of the numerical calculations of Schlichting and Bussman (6) and Emmons and Leigh (4) for the flat plate and those of Eckert and Hartnett (2) for the stagnation flow. A surprising result is that even for $n = 0$ the asymptotic expression is not too badly in error, 24% for the stagnation flow and 51% for the flat plate, which suggests that a simple interpolation formula between the two asymptotes, $n = 0$ and $n \to 1$, could be used for the whole range. As can be seen from Figures 2 and 3 the expression

$$N_{Nu} = \left[(N_{Nu})_0^{\frac{3}{2}} + (n(N_{Nu})_1)^{\frac{3}{2}} \right]^{\frac{2}{3}} \tag{14}$$

is in excellent agreement with the exact results for both the flat plate and the stagnation flow, and it is suggested therefore that Equation (14) be used for all surface geometries. $(N_{Nu})_0$ is the Nusselt number in the absence of an interfacial velocity ($n = 0$); $(N_{Nu})_1$ is given by Equation (13).

Variable property fluids

The analysis has dealt so far with a constant property fluid. For many physical systems however this may introduce a considerable error in the results, especially in the case where $W_s << W_\infty$ and $n \to 1$ owing to the large difference in the composition between the bulk and the region near the surface. This section, then, shows how problems involving a variable property fluid may be attacked.

The mass density ρ and the viscosity μ are undoubtedly the properties most sensitive to composition; the coefficient of ordinary diffusion D is, to a first approximation, independent of composition. The following analysis will however be made as general as possible. The basic equations are (3)

$$\rho' \left[u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} \right] = U_i \frac{dU_i}{dx_i} + \frac{\partial}{\partial y_1} \left(\mu' \frac{\partial u_1}{\partial y_1} \right) \tag{15}$$

$$\frac{\partial}{\partial x_1} (\rho' u_1) + \frac{\partial}{\partial y_1} (\rho' v_1) = 0 \tag{16}$$

$$\rho' \left[u_1 \frac{\partial \theta}{\partial x_1} + v_1 \frac{\partial \theta}{\partial y_1} \right] = \frac{1}{N_{Sc}} \frac{\partial}{\partial y_1} \left(\rho' D' \frac{\partial \theta}{\partial y_1} \right) \tag{17}$$

with

$$v_1 = - \frac{n D'_s}{N_{Sc}} \left(\frac{\partial \theta}{\partial y_1} \right) \quad \text{at } y_1 = 0$$

where the symbols have the same meaning as before, except that the fluid properties are divided by their value in the bulk. Thus

$$N_{Re} \equiv \frac{U_\infty L}{\nu_\infty}, \; N_{Sc} = \frac{\nu_\infty}{D_\infty}$$

$$\rho' = \frac{\rho}{\rho_\infty}, \; D' = \frac{D}{D_\infty}, \; \mu' = \frac{\mu}{\mu_\infty}$$

and D'_s is the value of D' at the surface. The discussion will be limited to the case where $n \to 1$, since it is believed that once the asymptotic expression for the Nusselt number has been calculated, the complete solution may again be approximated by Equation (14) with satisfactory accuracy.

Once more, the transformation given by Equations (5) and (6) may be used to reduce the equations in the limit as $n \to 1$. It is easy to show that the expressions corresponding to Equations (11) become

$$\frac{d}{dz} \left(\mu' \frac{d^2 \phi}{dz^2} \right) + \frac{n}{N_{Sc}} \rho'_s D'_s \frac{d^2 \phi}{dz^2} = 0 \tag{18}$$

and

$$\frac{d}{dz} \left(\rho' D' \frac{d\theta}{dz} \right) + N_{Sc} \left[\frac{n}{N_{Sc}} D'_s \rho'_s + \frac{1}{b^2} {}_0\!\int^z \rho' \frac{d\phi}{dz} dz \right] \frac{d\theta}{dz} = 0 \tag{19}$$

$$\begin{aligned} \phi = \phi' = \theta = 0 \quad & \text{at } z = 0 \\ \phi' = \theta = 1 \quad & \text{at } z = \infty \\ \theta' = 1 \quad & \text{at } z = 0 \end{aligned}$$

This system may be solved in the limit $n \to 1$ by the following simple scheme. Equation (19) is first rearranged, after a transformation of variables from z to θ, into

$$\rho' D' \frac{d\theta}{dz} = \rho'_s D'_s - N_{Sc} \left[\frac{n\theta}{N_{Sc}} D'_s \rho'_s + \frac{1}{b^2} {}_0\!\int^\theta (\theta - \epsilon) \rho'(\epsilon) \frac{d\phi}{d\epsilon} d\epsilon \right] \tag{20}$$

However as $n \to 1$ ($b \to \infty$)

$$\frac{d\phi}{d\theta} \to \frac{D'\rho'}{D'_s \rho'_s (1-\theta)} \left[1 - e^{-\frac{\rho'_s D'_s}{N_{Sc}} \omega} \right] \tag{21}$$

where

$$\omega \equiv {}_0\!\int^\theta \frac{D'\rho'}{D'_s \rho'_s} \frac{d\epsilon}{\mu'(1-\epsilon)} \tag{21a}$$

and the eigenvalue problem, Equation (20), can therefore be solved by quadratures, since one of the boundary conditions requires that

$$\frac{d\theta}{dz} \to 0 \text{ as } \theta \to 1$$

The generalization to Equation (13) is then

$$N_{Nu} = \frac{U_i}{\sqrt{2(1-n) \, {}_0\!\int^{x_1} U_i dx}} \frac{\sqrt{N_{Re} \cdot N_{Sc}}}{(D'_s \rho'_s)} \sqrt{ {}_0\!\int^1 D'(\rho')^2 \left[1 - e^{\frac{\rho'_s D'_s \omega}{N_{Sc}}} \right] d\theta } \tag{22}$$

where D', ρ', and μ' are known functions of θ.

ASYMPTOTIC SOLUTION FOR FREE CONVECTION AS $n \to 1$

In free convection fluid motion is caused by the buoyancy forces and obeys, again for laminar–boundary-layer flows under isothermal conditions, the basic equations

$$u_2 \frac{\partial u_2}{\partial x_2} + v_2 \frac{\partial u_2}{\partial y_2} = (1 - \theta) G(x_2) + \frac{\partial^2 u_2}{\partial y_2^2} \tag{23}$$

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} = 0 \tag{24}$$

$$u_2 \frac{\partial \theta}{\partial x_2} + v_2 \frac{\partial \theta}{\partial y_2} = \frac{1}{N_{Sc}} \frac{\partial^2 \theta}{\partial y_2^2} \tag{25}$$

The boundary conditions are as before:

At $y_2 = 0$, $u_2 = 0$, $\theta = 0$,

and $v_2 = - \dfrac{n}{N_{Sc}} \dfrac{\partial \theta}{\partial y_2}$

At $x_2 = 0$ as well as

$$y_2 = \infty, \; u_2 = 0, \; \theta = 1 \tag{26}$$

It is again possible to arrive at an asymptotic solution for $n \to 1$ and for an arbitrary surface geometry. Let

$$\eta \equiv \left(\frac{3}{4} \right)^{1/4} \frac{y_2 G^{1/3}}{[{}_0\!\int^{x_2} G^{1/3} dx]^{1/4}} \tag{27}$$

and

$$u_2 = F'(\eta) \left(\frac{4}{3} \right)^{1/2} G^{1/3} \sqrt{{}_0\!\int^{x_2} G^{1/3} dx} \tag{28}$$

which transforms the original equations into

$$F''' + FF'' + (1-\theta) = (F')^2 \left(\frac{4}{3} \right)^{1/2} \frac{\sqrt{{}_0\!\int^{x_2} G^{1/3} dx}}{G^{2/3}} \frac{d}{dx_2} \left(\frac{4}{3} \right)^{1/2} G^{1/3} \sqrt{{}_0\!\int^{x_2} G^{1/3} dx} \tag{29}$$

and

$$\theta'' + N_{Sc} F\theta' = 0 \qquad (30)$$

with the boundary conditions

$$\text{At } \eta = 0, F = \frac{n}{N_{Sc}}\theta', F' = 0, \theta = 0$$

$$\text{At } \eta = \infty, F' = 0, \theta = 1 \qquad (31)$$

It is now obvious from Equation (29) that a similarity transformation cannot, strictly speaking, be assumed unless the surface geometry is such that

$$\frac{\sqrt{{}_0\int^{x_2} G^{1/3} dx}}{G^{2/3}} \frac{d}{dx_2} G^{1/3} \sqrt{{}_0\int^{x_2} G^{1/3} dx} = \text{a}$$

constant for all x_2 which is possible only if $G(x_2)$ is of the form

$$G(x_2) = x_2^m$$

It will be shown, however, that the transformation given by Equation (27) will apply to any surface as long as $n \rightarrow 1$. Let

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} \equiv b_1 \qquad (32)$$

where $b_1 \rightarrow \infty$ as $n \rightarrow 1$, and define

$$z_1 \equiv b_1 \eta \qquad (33a)$$

$$F \equiv \frac{n}{N_{Sc}} b_1 + \frac{\phi_1(z_1)}{b_1^3} \qquad (33b)$$

$$\theta \equiv \theta(z_1) \qquad (33c)$$

Therefore

$$\phi'''_1 + \frac{n}{N_{Sc}}\phi''_1 + (1-\theta)$$

$$= O\left(\frac{1}{b_1^4}\right) \rightarrow 0 \text{ as } n \rightarrow 1 \qquad (34)$$

and

$$\theta'' + N_{Sc}\left[\frac{n}{N_{Sc}} + \frac{\phi_1}{b_1^4}\right]\theta' = 0 \qquad (35)$$

with the boundary conditions

$$\text{At } \eta = 0, \phi_1 = \phi'_1 = 0, \theta = 0, \theta' = 1$$

$$\text{At } \eta = \infty, \phi'_1 = 0, \theta = 1$$

One then is dealing again with an eigenvalue problem, which fortunately may be solved by direct integration. Thus in the limit as $n \rightarrow 1$

$$b_1 \rightarrow \left[\frac{N_{Sc}^2}{2(1-n)[1+N_{Sc}]}\right]^{1/4}$$

so that

$$N_{Nu} = \left[\frac{3}{8(1-n)} \frac{N_{Gr}(N_{Sc})^2}{1+N_{Sc}}\right]^{1/4}$$

$$\frac{G^{1/3}}{\sqrt[4]{{}_0\int^{x_2} G^{1/3} dx}} \{1 + O(1-n)\} \qquad (36)$$

It can also be shown that under similar conditions the asymptotic form of the Nusselt number for a variable property fluid is

$$N_{Nu} = \left[\frac{3N_{Gr}N_{Sc}}{8(1-n)}\right]^{1/4} \frac{G^{1/3}}{\sqrt[4]{{}_0\int^{x_2} G^{1/3} dx}}$$

$$\sqrt[4]{\frac{{}_0\int^1 D'(\rho')^2 \Phi d\theta}{(D'_s\rho'_s)^2}} \qquad (37)$$

where

$$\Phi \equiv \frac{Z(\infty) N_{Sc}}{D'_s\rho'_s}\left[1 - e^{-\frac{\rho'_s D'_s}{N_{Sc}}\omega}\right.$$

$$\left. - {}_0\int^{\omega} e^{-\frac{\rho'_s D'_s}{N_{Sc}}(\omega - x)} Z(x) dx\right]$$

$$\omega \equiv {}_0\int^{\theta} \frac{D'\rho'}{D'_s\rho'_s} \frac{d\epsilon}{\mu'(1-\epsilon)}$$

and

$$Z(\omega) \equiv {}_0\int^{\theta(\omega)} \beta'(\theta) \frac{D'\rho'}{D'_s\rho'_s} d\theta;$$

$$Z(\infty) = {}_0\int^{1} \beta'(\theta) \frac{D'\rho'}{D'_s\rho'_s} d\theta$$

Again ρ', D', μ', and β' are the appropriate fluid properties divided by their value in the bulk.

Interestingly enough, the asymptotic expression for the Nusselt number, Equation (36), seems to hold with fair accuracy even for $n \sim 0$, exactly as was the case with the analogous forced-convection formula, Equation (13). For the vertical flat plate, for example, $G(x_1) = 1$, and

$$\frac{Nu}{(N_{Gr})^{1/4}} = 0.401 \text{ for } n = 0 \text{ and } N_{Sc} = 1$$

from reference 5 as compared with 0.66, which is predicted from Equation (36).

CONCLUSIONS

Asymptotic expressions were developed in this paper for the mass transfer Nusselt number in both forced- and free-convection laminar–boundary-layer flows with large interfacial velocities directed toward the surface. The analysis is valid for arbitrary surface geometries and includes transfer in a variable property fluid. It was shown in addition how these asymptotic formulas may be used in conjunction with the Nusselt number for zero interfacial velocities to estimate the rate of mass transfer for the intermediate regions.

The present development holds however, only for laminar–boundary-layer flows. This requires that in free convection $N_{Gr} >> 1$, and in forced convection $N_{Re} >> 1$; also, since it is assumed that the velocity distribution in the boundary layer does not affect the potential flow solution, it requires that

$$\frac{1}{N_{Re}(1-n)} << 1$$

which may be satisfied if N_{Re} is large enough.

ACKNOWLEDGMENT

This research was supported in part by a grant from the National Science Foundation and the Petroleum Research Fund administered by the American Chemical Society. Grateful acknowledgment is hereby made to the donors of these funds.

NOTATION

D	= diffusion coefficient
L	= characteristic length of the surface
u	= x component of the velocity vector
u_1	= u/U_∞
u_2	= $uL/\nu\ (1/N_{Gr})^{1/2}$
U_∞	= characteristic velocity of the bulk
U_i	= potential flow distribution outside the boundary layer, divided by U_∞
v	= y component of the velocity vector
v_1	= $v/U_\infty \sqrt{N_{Re}}$
v_2	= $vL/\nu\ (1/N_{Gr})^{1/4}$
x	= distance along the surface from the leading edge
x_1, x_2	= x/L
y	= distance along the surface from the leading edge
y_1	= $y/L \sqrt{N_{Re}}$
y_2	= $y/\text{L}\ (N_{Gr})^{1/4}$
W	= weight fraction of the diffusing species
W_s, W_∞	= weight fraction respectively at the surface and in the bulk
N_{Gr}	= Grashof number, $\beta g L^3/\nu^2$
N_{Nu}	= Nusselt number, $-L(\partial\theta/\partial y)_{y=0}$
N_{Re}	= Reynolds number, $U_\infty L/\nu$
N_{Sc}	= Schmidt Number, ν/D

Greek Letters

β	= expansion coefficient for the fluid which is defined by $\rho_\infty/\rho = 1 + \beta(1-\theta)(W_\infty - W_s)$
θ	$\equiv W - W_s/W_\infty - W_s$
ν	= kinematic viscosity
ρ	= mass density of the fluid
μ	= viscosity

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Manuscript received June 12, 1959; revision September 9, 1959; paper accepted September 14, 1959.